with

$$C(I) = \lambda\left(I - \frac{1}{2}\right) + \frac{1 - 12(I - \frac{1}{2})^2}{(I - \frac{1}{2})}$$

$$B(I) = 10\lambda(I - 1) + \frac{2(I - 1)[12(I - \frac{1}{2})(I - \frac{3}{2}) - 1]}{(I - \frac{1}{2})(I - \frac{3}{2})} \quad \text{(A4)}$$

and

$$A(I) = C(I - 1)$$
$$F(I) = -C(I) + \lambda(2I - 1)$$
$$E(I) = -B(I) + 20\lambda(I - 1)$$
$$D(I) = F(I - 1).$$

The first boundary condition (10) is satisfied with

$$C(M - 1) = F(M - 1) = 0. \quad \text{(A5)}$$

This set of equations is valid for $I = 2, \cdots M - 1$. For $I = 1$, in order to include the second boundary condition, we use expansions. We get

$$A(1) = D(1) = 0$$
$$B(1) = \lambda + 4$$
$$C(1) = -4$$
$$E(1) = \lambda - 4$$
$$F(1) = 4. \quad \text{(A6)}$$

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Modes and Cutoff Frequencies of Crossed Rectangular Waveguides

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***Abstract*—One complete solution is presented for determining the electromagnetic field of a generalized crossed rectangular waveguide. The method adopted is that of partial regions. Cutoff frequencies of symmetrical crossed waveguides are presented as an example. The results, even for low-order approximations, correspond well with the only experimental results available in the literature.**

I. Introduction

PRACTICAL waveguides usually have rectangular or circular cross sections whose cutoff frequencies and field equations have been known for years through the method of separation of variables. Other cross-sectional shapes are possible, but in general few of these have been investigated. Recently, crossed rectangular waveguide shapes have been of interest due to the fact that they may offer some advantages in terms of circular polarization and wider bandwidth or both [1]. Moreover, it has been shown experimentally that a dichroic panel of symmetrical crossed rectangular slots offers some favorable results as far as bandwidth and transmission at oblique angles of incidence are concerned [2].

The knowledge of crossed rectangular waveguides is, to the best knowledge of the author, limited to the paper by Stalzer *et al.* [1] who use a computer program developed by Konrad and Silvester [3] to calculate cutoff frequencies and field patterns. They also performed experimental work on the measurement of individual modes. Mathematical expressions for fields inside a crossed rectangular waveguide have not been reported.

Mathematical expressions for the fields within the guide are important. For example, if one is interested in modal matching techniques [4], [5] to give predictions for transmission of electromagnetic waves through a thin conducting screen periodically perforated with crossed rectangular slots, one has to know these relations.

The intent of this report is to find cutoff frequencies and expressions for the fields within the waveguide. Cutoff frequencies of symmetrical crossed waveguide shapes are presented in the form of graphs. The results, even for a 3×3 approximation, correspond well with the experimental results reported [1].

The generalized crossed rectangular waveguide studied in

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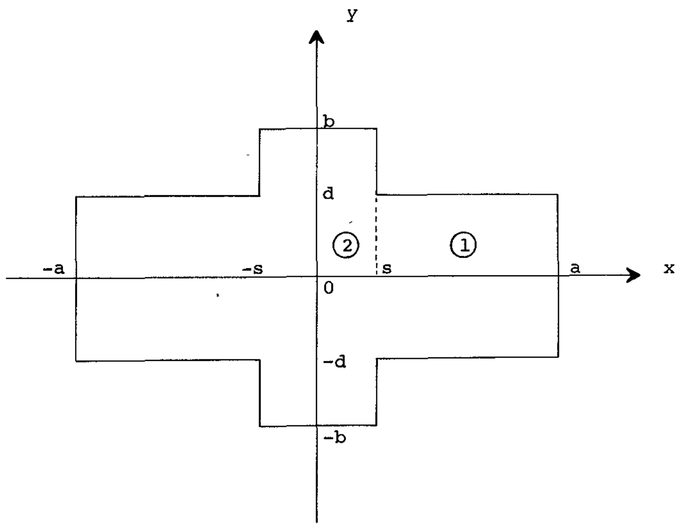

Fig. 1. Generalized crossed rectangular waveguide.

this paper is shown in Fig. 1. Since the structure is symmetrical to both the x and y axes, only the first quadrant has to be analyzed. In what follows, reference will be made to $TE_{m,n}$ and $TM_{m,n}$ modes to conform with standard notation normally used in the literature [6]. Boundary conditions on the x and y axes will determine the nomenclature of the modes; i.e., $TM_{odd,odd}$ and $TE_{odd,odd}$ are referred to modes with the x and y axes as magnetic walls; $TM_{odd,even}$ and $TE_{odd,even}$ to modes with the x axis as the electric wall, and the y axis as the magnetic wall; $TM_{even,odd}$ and $TE_{even,odd}$, to modes with the x axis as the magnetic wall and the y axis as the electric wall; $TM_{even,even}$ and $TE_{even,even}$ to modes with the x and y axes as electric walls.

The solution to the problem is accomplished by the partial region method used by Butcher [7] as well as Collins and Daly [8]. This less well-known method has also been used to study modes of rectangular coaxial waveguides [9].

The method, together with the derivation of $TM_{odd,odd}$ modes, will be described in Section II. Results for all other modes are presented in the Appendix.

II. Derivation

In order to analyze the $TM_{odd,odd}$ modes, it is only necessary to solve the wave equation for the first quadrant of Fig. 1, subjected to the appropriate boundary conditions. From here on, we will use subscripts 1 and 2 to refer to the regions 1 and 2 indicated in Fig. 1.

Let us denote the E field along the z axis (with $e^{j\omega t}$ omitted) as

$$E_{z1} = \sum_{r=1,3,\cdots}^{\infty} \phi_{1r} \sinh\left[p_{1r}\left(\frac{x}{2a} - \frac{1}{2}\right)\right] \sin\frac{r\pi(d-y)}{2d} \quad (1)$$

$$E_{2z} = \sum_{m=1,3,\cdots}^{\infty} \phi_{2m} \cosh\left(p_{2m}\frac{x}{2a}\right) \sin\frac{m\pi(b-y)}{2b} \quad (2)$$

where

$$p_{1r}^2 = -4k^2a^2 + \left(\frac{r\pi a}{d}\right)^2 \quad (3)$$

$$p_{2m}^2 = -4k^2a^2 + \left(\frac{m\pi a}{b}\right)^2 \quad (4)$$

and $k^2 = \omega^2\mu\varepsilon - \beta^2$; β is the propagation constant, and the guide cutoff wavelength λ_c is equal to $2\pi/k$.

Obviously, all the boundary conditions, including the conditions that the x and y axes are magnetic walls, are satisfied except along the boundary of the two regions as divided. The whole problem then becomes that of matching conditions at the boundary and then determining the constants ϕ_{1r} and ϕ_{2m} of (1) and (2).

Continuity of E_z along the boundary implies that $E_{z1} = E_{z2}$ at $x = s$. When the equation is multiplied by $\sin(n\pi(b-y)/2b), n = 1,3,\cdots$ on both sides and integrated from zero to b, and noting $E_{z1} = 0$ for $x = s$ and $d \leq y \leq b$, we find that

$$\tfrac{1}{4}\phi_{2n}\cosh\left(p_{2n}\frac{s}{2a}\right) = \sum_{r=1,3,\cdots}^{\infty} \phi_{1r}\sinh\left[p_{1r}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] K_{rn} \quad (5)$$

where

$$K_{mn} = \frac{1}{2b}\int_0^d \sin\frac{m\pi(d-y)}{2d}\sin\frac{n\pi(b-y)}{2b}\,dy. \quad (6)$$

Similarly, continuity of H_y along the boundary at $x = s$ requires that $\partial E_{z1}/\partial x$ be equated to $\partial E_{z2}/\partial x$; when the resulting equation is multiplied by $\sin[j\pi(d-y)/2d]$, $j = 1,3,\cdots$ on both sides, and both sides are then integrated from zero to d, we find that

$$\frac{d\phi_{1j}p_{1j}}{2}\cosh\left[p_{1j}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] = \sum_{m=1,3,\cdots}^{\infty} 2bK_{jm}\phi_{2m}p_{2m}\sinh\left(p_{2m}\frac{s}{2a}\right). \quad (7)$$

Eliminating ϕ_{1j} from (5) and (7), we have

$$\sum_{n=1,3,\cdots}^{\infty}\sum_{m=1,3,\cdots}^{\infty} \phi_{2m}a_{nm} = 0 \quad (8)$$

where

$$a_{nm} = \sinh\left(p_{2m}\frac{s}{2a}\right)\left\{\frac{16bp_{2m}}{d}\sum_{r=1,3,\cdots}^{\infty}\frac{K_{rn}K_{rm}}{p_{1r}} \cdot \tanh\left[p_{1r}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] - \frac{\delta_{nm}}{\tanh[p_{2m}(s/2a)]}\right\} \quad (9)$$

and

$$\delta_{nm} = \begin{cases}1, & m = n\\ 0, & \text{otherwise.}\end{cases} \quad (10)$$

Equation (8) will have a nontrivial solution for ϕ_{2m} if and only if

$$\det[a_{nm}] = 0. \quad (11)$$

So by solving (11), we can locate cutoff frequencies corresponding to $TM_{odd,odd}$ modes; then ϕ_{2m} can be obtained for all m. ϕ_{1j} for all j can be deduced from (7). Once E_{z1} and E_{z2} are known, E_x, E_y, H_x, and H_y can be determined for all frequencies from Maxwell's equations.

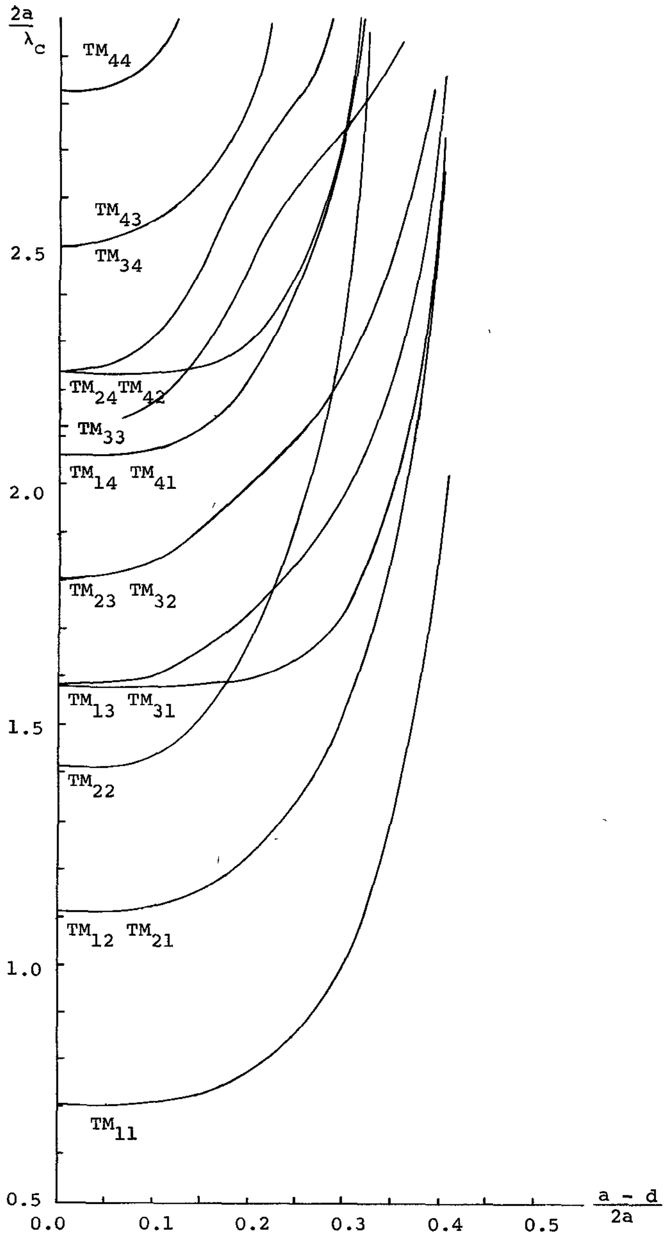

Fig. 2. Cutoff frequencies for TM modes of symmetrical crossed waveguide ($a = b$ and $d = s$).

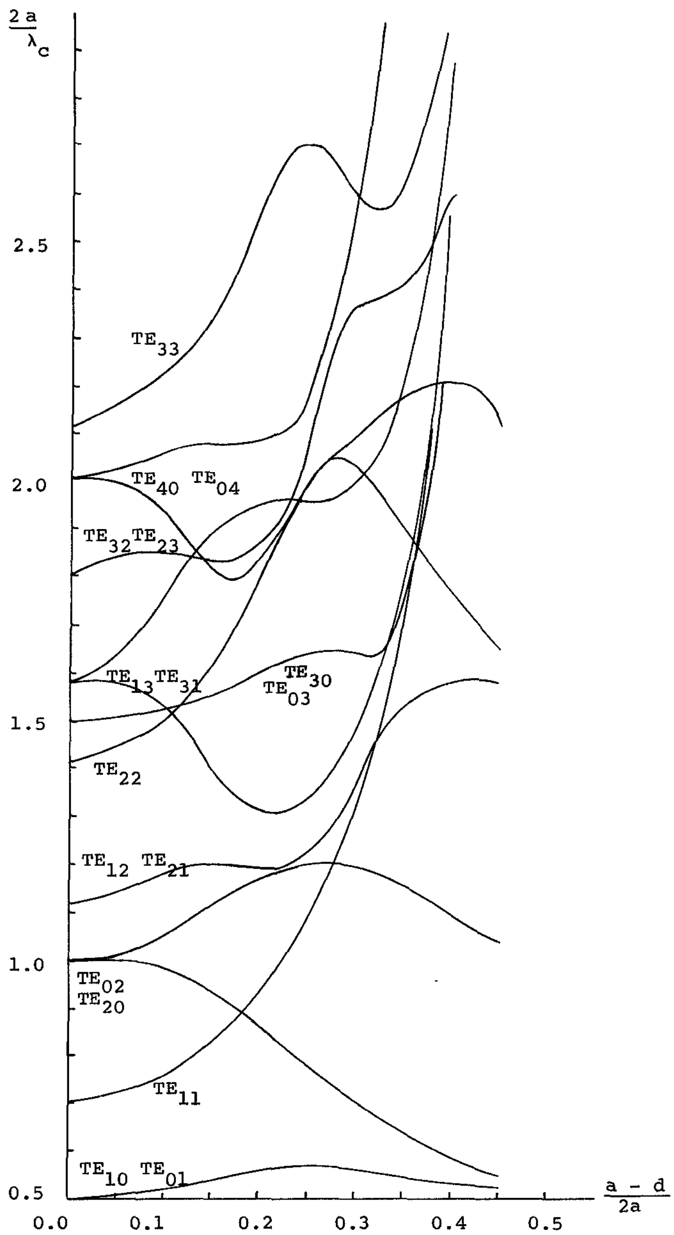

Fig. 3. Cutoff frequencies for TE modes of symmetrical crossed waveguide ($a = b$ and $d = s$).

III. Computation and Result

The solutions of (11) for the $TM_{odd,odd}$ mode and all other equations for other modes are obtained through low-order approximations. Cutoff frequencies are obtained by mapping the whole range of frequencies of interest. The solutions are determined as the images being equal to zero within the range. The function is a complex valued function. Care should be taken in interpreting and extrapolating the solutions at the image points. Final results are presented in Figs. 2 and 3.

It should be noted that the solution shows the only possible modes are those which correspond one-to-one with square waveguide modes. Notice the phenomena of the splitting of modes. By this method, it takes eight sets of equations (see Appendix) to generate all modes for crossed rectangular waveguides. This eliminates some of the confusion in finding solutions for a particular mode.

The only approximation involved is the number of terms in (1) and (2) to represent the fields. Therefore, the solution can be made as precise as needed. Due to computer time consideration, only a 3×3 approximation is used here. A 3×3 approximation of this technique has been reported to be sufficient to give reasonable result [9]. TE_{10}, TE_{01}, TE_{02}, TE_{20}, TE_{11}, and TM_{11} as calculated here are identical to the only available measurement data found in the literature [1].

IV. Conclusion

This report is for the general case of crossed rectangular waveguides. Cutoff frequencies for symmetrical crossed waveguides have been presented as an example. Field equations can therefore be generated. The same results have been obtained using the Ritz–Galerkin method [10].

Appendix

$TM_{odd,even}$ Modes: The transverse components of the fields, the equation relating the cutoff frequencies and field coefficients, and the relation between ϕ_{1j} and ϕ_{2m} are the same as those in Section II with the only exception that r,m,n are now nonzero even integers.

$TM_{even,odd}$ Modes: E_{z1} is given by (1). E_{z2} is given by (2) with cosh replaced by sinh. The equation relating the cutoff frequencies and field coefficients is

$$\sum_{n=1,3,\cdots}^{\infty} \sum_{m=1,3,\cdots}^{\infty} \phi_{2m} \cosh\left(p_{2m}\frac{s}{2a}\right) \cdot \left\{\frac{16bp_{2m}}{d} \sum_{r=1,3,\cdots}^{\infty} \frac{K_{rn}K_{rm}}{p_{1r}} \tanh\left[p_{1r}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] - \frac{\delta_{nm}}{\coth\left[p_{2m}(s/2a)\right]}\right\} = 0 \quad (12)$$

and ϕ_{1j} and ϕ_{2m} are related by

$$\frac{d\phi_{1j}p_{1j}}{2} \cosh\left[p_{1j}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] = \sum_{m=1,3,\cdots}^{\infty} 2bK_{jm}\phi_{2m}p_{2m} \cosh\left(p_{2m}\frac{s}{2a}\right). \quad (13)$$

$TM_{even,even}$ Modes: All relations are the same as those of $TM_{even,odd}$ modes with the only exception that r,m,n are nonzero even integers.

$TE_{odd,even}$ Modes: The transverse components of the magnetic field are given by

$$H_{z1} = \sum_{r=0,2,\cdots}^{\infty} \phi_{1r} \cosh\left[p_{1r}\left(\frac{x}{2a}-\frac{1}{2}\right)\right] \cos\frac{r\pi(d-y)}{2d} \quad (14)$$

$$H_{z2} = \sum_{m=0,2,\cdots}^{\infty} \phi_{2m} \sinh\left(p_{2m}\frac{x}{2a}\right) \cos\frac{m\pi(b-y)}{2b}. \quad (15)$$

The equation relating the cutoff frequencies and field components is

$$\sum_{n=0,2,\cdots}^{\infty} \sum_{m=0,2,\cdots}^{\infty} \phi_{2m} \sinh\left(p_{2m}\frac{s}{2a}\right) \cdot \left\{\frac{4d}{b} \sum_{r=0,2,\cdots}^{\infty} p_{1r}\frac{C_{rn}C_{rm}}{\Delta_r} \tanh\left[p_{1r}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] - \frac{p_{2m}\Delta_m\delta_{nm}}{\tanh\left[p_{2m}(s/2a)\right]}\right\} = 0 \quad (16)$$

where Δ_r is defined as

$$\Delta_r = \begin{cases} 2, & r = 0 \\ 1, & \text{otherwise.} \end{cases} \quad (17)$$

C_{mn} is defined as

$$C_{mn} = \frac{1}{d}\int_0^d \cos\frac{m\pi(d-y)}{2d} \cos\frac{n\pi(b-y)}{2b}\, dy. \quad (18)$$

ϕ_{1m} and ϕ_{2n} are related by

$$\frac{\phi_{1m}}{2}\Delta_m \cosh\left[p_{1m}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] = \sum_{n=0,2,\cdots}^{\infty} \phi_{2n}C_{mn} \sinh\left(p_{2n}\frac{s}{2a}\right). \quad (19)$$

$TE_{odd,odd}$ Modes: All relations of those of $TE_{odd,even}$ hold with the only exception that r,m,n are odd integers.

$TE_{even,even}$ Modes: H_{z1} is given by (14). H_{z2} is given by (15) with sinh replaced with cosh. The equation relating the cutoff frequencies and field coefficients is

$$\sum_{n=0,2,\cdots}^{\infty} \sum_{m=0,2,\cdots}^{\infty} \phi_{2m} \cosh\left(p_{2m}\frac{s}{2a}\right) \cdot \left\{\frac{4d}{b} \sum_{r=0,2,\cdots}^{\infty} p_{1r}\frac{C_{rn}C_{rm}}{\Delta_r} \tanh\left[p_{1r}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] - \frac{\Delta_m p_{2m}\delta_{nm}}{\coth\left[p_{2m}(s/2a]\right)}\right\} = 0 \quad (20)$$

and ϕ_{1m} and ϕ_{2n} are related by

$$\frac{\phi_{1m}}{2}\Delta_m \cosh\left[p_{1m}\left(\frac{s}{2a}-\frac{1}{2}\right)\right] = \sum_{n=0,2,\cdots}^{\infty} \phi_{2n}C_{mn} \cosh\left(p_{2n}\frac{s}{2a}\right). \quad (21)$$

Δ_m, C_{rm}, and δ_{mn} are all defined earlier.

$TE_{even,odd}$ Modes: All relations are the same as those of $TE_{even,even}$ modes with the only exception that r,m,n are odd integers.

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